Comparing a series to an integral
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We consider the difference between the definite integral

\[ \int_{0}^{\infty} u^x e^{-u} \, du, \]

where \( x \) is a real parameter, and the approximating sum \( \sum_{k=1}^{\infty} k^x e^{-k} \). We use properties of Bernoulli numbers to show that this difference is unbounded and has infinitely many zeros. We also conjecture that the sign of the difference at any positive integer \( n \) is determined by the sign of \( \cos((n + 1) \arctan(2\pi)) \).

1. Introduction

There are a variety of situations where it is necessary to examine differences of sums and integrals. The Euler–Maclaurin summation formula is the usual tool for estimating \( \int_{0}^{Y} g(u) \, du - \sum_{n \leq Y} g(n) \) [Abramowitz and Stegun 1964, p. 806], but it can also be interesting to develop exact formulas for particular choices of \( g(u) \). For instance, the Euler–Mascheroni constant arises if we set \( g(u) = 1/u \) and consider the limit as \( Y \to \infty \) [Wells 1986, p. 12]. The purpose of this paper is to examine the function

\[ f(x) := \sum_{k=1}^{\infty} k^x e^{-k} - \int_{0}^{\infty} t^x e^{-t} \, dt. \]

The integral on the right equals \( \Gamma(x + 1) \), where \( \Gamma(x) \) is the gamma function, and the infinite series converges absolutely for all values of \( x \). We can obtain an exact expression for \( f(n) \) when \( n \geq 1 \) by using classical formulas for polylogarithms of negative order [Weisstein 2013]:

\[ f(n) = -n! + \sum_{k=0}^{n} \frac{1}{(e - 1)^{k+1}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k + 1 - j)^n. \]  

(1)

The main goal of this paper is to prove that \( f(x) \) has infinitely many positive real zeros, and that the function becomes unbounded as \( x \to \infty \). Further, in Conjecture 1 we hypothesize that \( f(n) \) has the same sign as \( \cos((n + 1) \arctan(2\pi)) \) whenever

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\( n \) is a positive integer. We prove that the conjecture is true with finitely many exceptions, provided that \( \arctan \left( \frac{2\pi}{\pi} \right) \) has finite irrationality measure. If we expand \( \cos \left( (n+1) \arctan \left( \frac{2\pi}{\pi} \right) \right) \) using trigonometric identities, then we obtain the equivalent conjecture that the following identity holds for all positive integers \( n \):

\[
\text{sign} \left[ \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{2j} (2\pi)^{2j} \right]
= \text{sign} \left[ -(e-1)^{n+1}n! + \sum_{k=0}^{n} (e-1)^{n-k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+1-j)^n \right]. \tag{2}
\]

The left-hand side of (2) is a polynomial in \( \pi \), while the right-hand side is a polynomial in \( e \). Based on numerical experiments, we conjecture that \((\pi, e)\) is the unique, nontrivial (i.e., \( \neq (0, 1) \)) tuple of real numbers which makes (2) valid for all positive integers \( n \). When we choose values close to \( \pi \) and \( e \) respectively, we notice that (2) is false for some \( n \) in all considered cases. Surprisingly, (2) is valid for \( n \leq 128 \) if you insert \((\pi + 0.015, e)\), but only for \( n \leq 2 \) in the case of \((\pi, e + 0.015)\). So the equation seems to be a lot more sensitive to small modifications in the argument on the right-hand side. Also, choosing various random tuples \((x, y)\) further away from \((\pi, e)\), we always found an \( n \) such that (2) was wrong.

2. Elementary properties of \( f(x) \)

In this section we prove that \( f(x) \) is an unbounded function by showing that the sequence \( \{f(n)\}_{n=1}^{\infty} \) is unbounded as \( n \to \infty \). Our proof uses properties of Bernoulli numbers. The \( n \)-th Bernoulli number is defined by

\[
\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{3}
\]

and the generating series converges for \( |x| < 2\pi \). It is known that Bernoulli numbers are always rational, and that \( B_n = 0 \) if \( n > 1 \) is odd. Bernoulli numbers have many interesting combinatorial properties [Abramowitz and Stegun 1964], and the following asymptotic holds for large values of \( n \):

\[
|B_{2n}| \sim \frac{n^{2n}}{(\pi e)^{2n}}. \tag{4}
\]

This property will be used later. We begin by deriving a new formula for \( B_n \). Then in Theorem 1, we use our formula to prove that \( f(x) \) is unbounded.

**Lemma 1.**

\[
B_n = \sum_{k=n}^{\infty} \frac{f(k) - kf(k-1)}{(k-n)!} \quad \text{for } n \geq 2.
\]
Proof. Consider the generating function of the Bernoulli numbers,

\[ g(x) := \frac{x}{e^x - 1}, \]

whose Taylor series at \( x = -1 \) is

\[ g(x) = \frac{e}{e - 1} - \frac{e(e - 2)}{(e - 1)^2} (x + 1) + \sum_{n=2}^{\infty} \frac{f(n) - nf(n-1)}{n!} (x + 1)^n. \]  \( (5) \)

The Taylor coefficients at \( n = 0 \) and \( n = 1 \) are calculated directly. To obtain the coefficients when \( n \geq 2 \), we use

\[ g^{(n)}(-1) = \frac{d^n}{dx^n} \left[ \frac{-x}{1 - e^x} \right]_{x=-1} = \frac{d^n}{dx^n} \left[ -x \sum_{m=0}^{\infty} e^{mx} \right]_{x=-1} = \sum_{m=1}^{\infty} m^n e^{-m} - n \sum_{m=1}^{\infty} m^{n-1} e^{-m} = \left( \sum_{m=1}^{\infty} m^n e^{-m} - n! \right) - n \left( \sum_{m=1}^{\infty} m^{n-1} e^{-m} - (n-1)! \right) = f(n) - nf(n-1). \]  \( (6) \)

Since formula (3) is also valid when \( x \) lies in a neighborhood of \(-1\), we can equate the two results:

\[ g(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(-1)}{n!} (x + 1)^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(-1)}{n!} \left( \sum_{k=0}^{n} \binom{n}{k} x^k \right) = \sum_{n=0}^{\infty} \left[ \sum_{k=n}^{\infty} \frac{g^{(k)}(-1)}{(k-n)!} \right] \frac{x^n}{n!}. \]

Comparing coefficients and then applying (6), we find that for \( n \geq 2 \),

\[ B_n = \sum_{k=n}^{\infty} \frac{g^{(k)}(-1)}{(k-n)!} = \sum_{k=n}^{\infty} \frac{f(k) - kf(k-1)}{(k-n)!}. \]  \( \square \)

Theorem 1. The sequence \( \{ f(n) \}_{n=1}^{\infty} \) is unbounded.

Proof. We construct a proof by contradiction. Assume that \( |f(n)| < C \) for some \( C > 0 \) and every \( n \in \mathbb{N} \). By Lemma 1 and the triangle inequality, we have

\[ |B_n| \leq \sum_{k=n}^{\infty} \frac{|f(k) - kf(k-1)|}{(k-n)!} \leq \sum_{k=n}^{\infty} \frac{C(1+k)}{(k-n)!} \leq Ce(n+2). \]

This contradicts the asymptotic \( |B_{2n}| \sim n^{2n}/(\pi e)^{2n} \), which holds for \( n \) sufficiently large.  \( \square \)
Remark. Despite the fact that $f(n)$ is unbounded as $n \to \infty$, the ratio $f(n)/n!$ converges to zero. To prove this, we can use residue calculus to show that
\[
\frac{f(n)}{n!} = \frac{1}{2\pi i} \oint_{\gamma} \frac{z^{n-1}}{1-e^z} \, dz,
\]
where $\gamma = \{ z \in \mathbb{C} : |z| = 2 \}$. We then employ the triangle inequality and numerical integration to obtain the crude upper bound $|f(n)/n!| \leq 0.82 \times 2^{-n}$. In fact, it is possible to develop a much sharper upper bound using formula (12) below.

**Theorem 2.** The function $f(x)$ has infinitely many zeros.

**Proof.** First notice that $f(2) \approx -0.0077$ and $f(3) \approx 0.0065$, so by continuity $f(x)$ has at least one zero in the interval $(2, 3)$. To prove that the function has infinitely many zeros, we proceed by contradiction.

Assume that $f$ has only finitely many zeros. Then for any sufficiently large integer $m$, the elements of the set \{ $f(m)$, $f(m+1)$, $f(m+2)$, \ldots \} all have the same sign. Now consider the function
\[
h(x) := \frac{1}{x} - \frac{1}{e^x - 1},
\]
which has the Taylor series
\[
h(x) = \frac{1}{e - 1} + \sum_{k=1}^{\infty} \frac{f(k)}{k!} (x + 1)^k. \tag{7}
\]
Differentiating $m$ times gives
\[
h^{(m)}(x) = \sum_{k=m}^{\infty} \frac{f(k)}{(k-m)!} (x + 1)^{k-m}. \tag{8}
\]
If the elements of the set \{ $f(m)$, $f(m+1)$, \ldots \} are strictly positive, then (8) becomes a sum over positive numbers whenever $x \in (-1, 0)$, and it follows that $h^{(m)}(x)$ is strictly positive. If we notice that
\[
h(x) = \frac{1}{x} - \frac{1}{x} \frac{x}{e^x - 1} = -\sum_{n=1}^{\infty} \frac{B_n}{n!} x^{n-1},
\]
then we also have
\[
h^{(m)}(x) = -\sum_{n=m+1}^{\infty} \frac{B_n}{n!} \frac{(n-1)!}{(n-m-1)!} x^{n-m-1}. \tag{9}
\]
The key observation is that formulas (8) and (9) have overlapping domains of convergence on the negative real axis near the origin. If $x$ is a sufficiently small
negative real number, then (9) implies
\[ h^{(m)}(x) \approx -\frac{B_{m+1}}{m+1}, \]
but (8) guarantees
\[ h^{(m)}(x) > 0. \]
This is a contradiction, because Bernoulli numbers assume both positive and negative values as \( m \) increases. We can deal with the case where \( \{ f(m), f(m+1), \ldots \} \) are strictly negative in a similar manner. \( \square \)

In Theorem 2 we proved that \( f(x) \) has infinitely many real zeros. In fact, we can be much more precise about the locations of the zeros. If \( x_j \) denotes the \( j \)-th positive real zero of \( f(x) \) such that \( f(x_j) = 0 \), then we expect that
\[ x_j \approx -1 + \frac{\pi(2j + 1)}{2 \arctan(2\pi)}. \] (10)
The first approximation gives \( x_1 \approx 2.335 \ldots, \) and this is reasonably close to the true value \( x_1 = 2.306 \ldots \) We have observed numerically that the approximations become more accurate for large values of \( j \). To derive (10), consider an identity which is valid for \( \text{Re}(x) > 0 \) and \( \text{Re}(\mu) > 0 \):
\[
\frac{1}{\Gamma(x+1)} \sum_{k=1}^{\infty} k^x e^{-\mu k} = \sum_{k=-\infty}^{\infty} \frac{1}{(\mu + 2\pi ik)^{x+1}}. \] (11)
Formula (11) is a special case of an identity due to Lipschitz [Rademacher 1973, p. 77], and follows from the Poisson summation formula. Set \( \mu = 1 \) and take the real part of both sides to obtain
\[
\frac{f(x)}{\Gamma(x+1)} = 2 \sum_{k=1}^{\infty} \frac{\cos((x+1) \arctan(2\pi k))}{(1 + 4\pi^2 k^2)^{(x+1)/2}}. \] (12)
Equation (12) converges rapidly, and we can approximate \( f(x) \) by truncating the series. The first term gives
\[
\frac{f(x)}{\Gamma(x+1)} \approx 2 \frac{\cos((x+1) \arctan(2\pi))}{(1 + 4\pi^2)^{(x+1)/2}}. \] (13)
and we immediately recover (10). It is somewhat subtle to determine how often (13) actually provides a good approximation of \( f(x) \), and we touch on this point in the next section.
3. A conjecture on the sign of $f(n)$

A second observation from (13) is that the sign of $f(n)$ should always equal the sign of $\cos((n + 1) \arctan(2\pi))$. We have verified this numerically for $n \leq 5000$ in Maple, and as a result we have the following conjecture:

**Conjecture 1.** For all positive integers $n$,

$$\text{sign } f(n) = \text{sign } \cos((n + 1) \arctan(2\pi)).$$

(14)

Equivalently, for every positive integer $n$,

$$\text{sign} \left[ \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{2j} (2\pi)^{2j} \right] = \text{sign} \left[ -(e-1)^{n+1} n! + \sum_{k=0}^{n} (e-1)^{n-k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+1-j)^n \right].$$

(15)

Conjecture 1 is easy to check numerically. The main difficulty in actually proving the conjecture is to determine how often (13) leads to a good approximation of $f(n)$. The reason that (14) might fail is because $(n + 1) \arctan(2\pi)$ is unreasonably close to a half-integer multiple of $\pi$. This would cause the first term of the infinite series in (12) to nearly vanish, in which case higher-order terms would dominate and the estimate in (13) would fail. Thus we need to rule out the possibility that $(n + 1) \arctan(2\pi)$ is unreasonably close to a half-integer multiple of $\pi$. This is equivalent to ruling out the possibility that $\arctan(2\pi)/\pi$ is unreasonably well approximated by rational numbers. Before proceeding, we note that $\arctan(2\pi)/\pi$ is trivially irrational, because otherwise we would have an identity of the form $2\pi = \tan(p\pi/q)$ for some $(p, q) \in \mathbb{Z}^2$, contradicting the transcendency of $\pi$.

**Lemma 2.** Equation (14) is true for any positive integer $n$ which satisfies

$$|\cos((n + 1) \arctan(2\pi))| > \frac{2.6}{1.98^{n+1}}.$$  

(16)

**Proof.** First, rewrite (12) as

$$\frac{f(n)}{n!} = 2 \frac{\cos((n + 1) \arctan(2\pi))}{(1 + 4\pi^2)^{(n+1)/2}} + 2 \sum_{k=2}^{\infty} \frac{\cos((n + 1) \arctan(2\pi k))}{(1 + 4\pi^2 k^2)^{(n+1)/2}}.$$

If the first term on the right dominates, then it follows easily that

$$\text{sign } \frac{f(n)}{n!} = \text{sign } \frac{2 \cos((n + 1) \arctan(2\pi))}{(1 + 4\pi^2)^{(n+1)/2}}.$$
and this is equivalent to Conjecture 1. Thus we need to prove
\[
\left| 2 \cos\left( (n + 1) \arctan(2\pi) \right) \right| \geq \left| 2 \sum_{k=2}^{\infty} \frac{\cos((n + 1) \arctan(2\pi k))}{(1 + 4\pi^2 k^2)^{(n+1)/2}} \right|. \tag{17}
\]
Equation (16) easily implies that
\[
\left| 2 \cos\left( (n + 1) \arctan(2\pi) \right) \right| > \frac{5.2}{1.98^{n+1}(1 + 4\pi^2(n+1)/2)} > \frac{5.2}{12.59^{n+1}}. \tag{18}
\]
On the other hand, by the triangle inequality
\[
\left| 2 \sum_{k=2}^{\infty} \frac{\cos((n + 1) \arctan(2\pi k))}{(1 + 4\pi^2 k^2)^{(n+1)/2}} \right| \leq 2 \sum_{k=2}^{\infty} \frac{1}{(1 + 4\pi^2 k^2)^{(n+1)/2}}
\leq \frac{2}{(1 + 16\pi^2)^{(n-1)/2}} \sum_{k=2}^{\infty} \frac{1}{1 + 4\pi^2 k^2}
\leq \frac{5.2}{(1 + 16\pi^2)(n+1)/2} < \frac{5.2}{12.6^{n+1}}. \tag{19}
\]
Thus combining (19) and (18) shows that
\[
\left| 2 \cos\left( (n + 1) \arctan(2\pi) \right) \right| - \left| 2 \sum_{k=2}^{\infty} \frac{\cos((n + 1) \arctan(2\pi k))}{(1 + 4\pi^2 k^2)^{(n+1)/2}} \right| > \frac{5.2}{12.59^{n+1}} - \frac{5.2}{12.6^{n+1}} > 0,
\]
and (17) follows immediately. Therefore Conjecture 1 is true whenever \( n \) is a positive integer for which (16) holds. \( \square \)

It is typically very tricky to determine how well a particular number \( \theta \) can be approximated by rational numbers. We say that \( \theta \) has irrationality measure \( \mu \) if \( \mu \) is the smallest real number such that
\[
\left| \theta - \frac{p}{q} \right| > \frac{1}{q^\mu}
\]
for all but finitely many pairs \( (p, q) \in \mathbb{Z}^2 \) with \( q > 0 \). The Thue–Roth–Siegel theorem guarantees that \( \mu = 2 \) whenever \( \theta \) is algebraic and irrational [Roth 1955]. An easy consequence of this theorem is that \( \theta \) can never be algebraic and have irrationality measure greater than 2. The typical method for proving that particular numbers are transcendental is to construct infinite sequences of rational numbers which approximate them too well. Liouville gave the first examples of transcendental
numbers in 1851 [Niven 1956, p. 93]. He proved that numbers like
\[ \theta_0 = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} \]
are always transcendental. Notice that if we set \( p_N = \sum_{n=1}^{N} 10^{N!-n!} \) and \( q_N = 10^{N!} \), then it is easy to show that
\[ \left| \theta_0 - \frac{p_N}{q_N} \right| \leq \frac{2}{q_N^{N+1}}. \]

Given any \( k > 0 \), this allows us to construct infinite sequences of rational numbers so that \( |\theta_0 - p/q| < 1/q^k \). Numbers with this property are called Liouville numbers and are said to have infinite irrationality measure. While a simple counting argument shows that almost all numbers are irrational, the set of Liouville numbers has measure zero inside the irrational numbers. Irrational numbers typically have finite irrationality measures; it is known that \( \pi \) has irrationality measure at most 7.6063 [Salikhov 2008], and \( \log 2 \) has irrationality measure at most 3.57455391 [Marcovecchio 2009].

**Theorem 3.** Assume that \( \arctan(2\pi)/\pi \) has finite irrationality measure. Then Conjecture 1 is true for \( n \) sufficiently large.

**Proof.** Assume that (16) fails for some integer \( n \). Then we have
\[ \frac{2.6}{1.98^{n+1}} \geq \left| \cos((n+1) \arctan(2\pi)) \right| = \left| \sin((n+1) \arctan(2\pi) - \frac{\pi}{2} - \pi j) \right| \]
for any integer \( j \). Select \( j \) so that \( z \in [-\pi/2, \pi/2] \), where \( z \) is the argument of the sine function. Elementary estimates show that \( |\sin z| \geq 2|z|/\pi \). Thus
\[ \frac{2.6}{1.98^{n+1}} \geq \frac{2}{\pi} \left| (n+1) \arctan(2\pi) - \frac{\pi}{2} - \pi j \right|, \]
and rearranging gives
\[ \frac{1.3}{(n+1)1.98^{n+1}} \geq \left| \frac{\arctan(2\pi)}{\pi} - \frac{2j + 1}{2(n+1)} \right|. \tag{20} \]

If \( \arctan(2\pi)/\pi \) has finite irrationality measure, then (20) can only hold for finitely many values of \( n \). We conclude that (16) holds for \( n \) sufficiently large, which implies that Conjecture 1 is also true for \( n \) sufficiently large. \( \square \)

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