RYSHKOV DOMAINS OF REDUCTIVE ALGEBRAIC GROUPS

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Dedicated to Professor Ichiro Satake on his 85th birthday

Let $G$ be a connected reductive algebraic group defined over a number field $k$. In this paper, we introduce the Ryshkov domain $R$ for the arithmetical minimum function $m_Q$ defined from a height function associated to a maximal $k$-parabolic subgroup $Q$ of $G$. The domain $R$ is a $Q(k)$-invariant subset of the adele group $G(\mathbb{A})$. We show that a fundamental domain $\Omega$ for $Q(k)\backslash R$ yields a fundamental domain for $G(k)\backslash G(\mathbb{A})$. We also see that any local maximum of $m_Q$ is attained on the boundary of $\Omega$.

Introduction

Let $P_n$ be the cone of positive definite $n$ by $n$ real symmetric matrices, and let $m(A)$ be the arithmetical minimum $\min_{0 \neq x \in \mathbb{Z}^n} \langle xAx \rangle$ of $A \in P_n$. The function $f : A \mapsto m(A)/(\det A)^{1/n}$ on $P_n$ is called the Hermite invariant. Since the maximum of $f$ gives the Hermite constant $\gamma_n$ for dimension $n$, the determination of local maxima of $f$ is a fundamental problem of lattice sphere packings in Euclidean spaces and the arithmetic theory of quadratic forms. Voronoi’s theorem [1908, Théorème 17] states that $f$ attains a local maximum at a point $A$ if and only if $A$ is perfect and eutactic. Moreover, perfect forms play an essential role in Voronoi’s reduction theory of $P_n$ with respect to the action of $\text{GL}_n(\mathbb{Z})$ (see, e.g., [Martinet 2003] and [Schürmann 2009]). Ryshkov [1970] introduced a locally finite polyhedron $R(m)$ in $P_n$ defined by the condition $m(A) \geq 1$. It is not difficult to show that $A$ is perfect with $m(A) = 1$ if and only if $A$ is a vertex of the boundary of $R(m)$. In particular, any local maximum of the Hermite invariant $f$ is attained on the boundary of $R(m)$. In this sense, we can say that the Ryshkov polyhedron $R(m)$ is well matched with $f$.

Let $G$ be a connected isotropic reductive algebraic group defined over a number field $k$, and let $Q$ be a maximal $k$-parabolic subgroup of $G$. In previous papers [Watanabe 2000; 2003], we investigated a constant $\gamma(G, Q, k)$ as a generalization of Hermite’s constant $\gamma_n$. Precisely, the constant $\gamma(G, Q, k)$ is defined to be

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the maximum of the function $m_Q(g) = \min_{x \in Q(k) \setminus G(k)} H_Q(xg)$ on $G(k) \setminus G(\mathbb{A})^1$, where $H_Q$ denotes the height function associated to $Q$. To prove the existence of the maximum of $m_Q$, we used Borel and Harish-Chandra's reduction theory for the adele group $G(\mathbb{A})$ with respect to $G(k)$. However, a Siegel set in $G(\mathbb{A})$ is not well matched with $m_Q$ in a sense that one cannot obtain any information on locations of extreme points of $m_Q$ in a Siegel set.

The purpose of this paper is to construct a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(k)$ which is well matched with $m_Q$. We first consider an analog of the Ryshkov polyhedron. We set $X_Q(g) = \{ x \in Q(k) \setminus G(k) : m_Q(g) = H_Q(xg) \}$ for a given $g \in G(\mathbb{A})^1$. This is a finite subset of $Q(k) \setminus G(k)$ and is regarded as an analog of the set of minimal vectors of a positive definite real quadratic form. We define the domain $R(m_Q)$ as follows:

$$R(m_Q) = \{ g \in G(\mathbb{A})^1 : \bar{e} \in X_Q(g) \},$$

where $\bar{e}$ denotes the trivial class $Q(k)$ in $Q(k) \setminus G(k)$. The set $R(m_Q)$ is a left $Q(k)$-invariant closed set with nonempty interior. The interior of $R(m_Q)$ is just a subset $R_1$ consisting of $g \in R(m_Q)$ such that $X_Q(g)$ is the one-point set $\{ \bar{e} \}$. We denote by $R_1^-$ the closure of $R_1$ in $G(\mathbb{A})^1$. Both $R_1$ and $R_1^-$ are also left $Q(k)$-invariant. By Baer and Levi's theorem [1931, Satz 7], there exists an open fundamental domain $\Omega_Q$ of $R_1^-$ with respect to $Q(k)$, that is, $\Omega_Q$ is a relatively open subset of $R_1^-$ satisfying

- $Q(k)\Omega_Q^- = R_1^-$, where $\Omega_Q^-$ denotes the closure of $\Omega_Q$ in $R_1^-$, and
- $\gamma \Omega_Q \cap \Omega_Q^- = \emptyset$ for any $\gamma \in Q(k) \setminus \{ e \}$.

Let $\Omega_Q^\circ$ denote the interior of $\Omega_Q$ in $G(\mathbb{A})^1$. Then our main theorem is stated as follows:

**Theorem.** The set $\Omega_Q^\circ$ is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(k)$. Any local maximum of $m_Q$ is attained on the intersection of the boundary of $\Omega_Q^\circ$ and the boundary of $R_1^-$. 

If we denote by $r_G$ the $k$-rank of the commutator subgroup of $G$, then $G$ has $r_G$ standard maximal $k$-parabolic subgroups. Since $\Omega_Q$ depends on $Q$, we obtain $r_G$ different kinds of fundamental domains of $G(\mathbb{A})^1$ with respect to $G(k)$. The method to construct $\Omega_Q$ may be viewed as a generalization of the highest point method (see [Grenier 1988] and [Terras 1988, §4.4]). For example, let $k = \mathbb{Q}$, $G = \text{GL}_n$ and $Q$ be a standard maximal $\mathbb{Q}$-parabolic subgroup such that $Q \setminus G$ is a projective space. Then our construction gives a fundamental domain $\Omega_Q$ whose Archimedean part is isomorphic with Grenier's fundamental domain. If we choose another standard maximal $\mathbb{Q}$-parabolic subgroup of $\text{GL}_n$ as $Q$, then the
Archimedean part of $\Omega_Q$ yields a new kind of fundamental domain of $\mathbb{P}_n$ with respect to $\text{GL}_n(\mathbb{Z})$ (see Example 3 in Section 7).

**Notation.** For a given ring $\mathfrak{A}$, the set of all $n$ by $k$ matrices with entries in $\mathfrak{A}$ is denoted by $M_{n,k}(\mathfrak{A})$. We write $M_n(\mathfrak{A})$ for $M_{n,n}(\mathfrak{A})$. The transpose of a given matrix $a \in M_{n,k}(\mathfrak{A})$ is denoted by $t^\prime a$. In this paper, $k$ denotes an algebraic number field of finite degree over $\mathbb{Q}$ and $\mathfrak{o}$ the ring of integers of $k$. The sets of all infinite and finite places of $k$ are denoted by $p_\infty$ and $p_f$, respectively. For $\sigma \in p_\infty \cup p_f$, $k_\sigma$ denotes the completion of $k$ at $\sigma$. For $\sigma \in p_f$, $\sigma_\sigma$ denotes the closure of $\mathfrak{o}$ in $k_\sigma$. The étale $\mathbb{R}$-algebra $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$ is identified with $\prod_{\sigma \in p_\infty} k_\sigma$. Let $\mathbb{A}$ and $\mathbb{A}^\times$ denote the adele ring and the idèle group of $k$, respectively. The idèle norm of $\mathbb{A}^\times$ is denoted by $| \cdot |_\mathbb{A}$.

1. Height functions

Let $G$ be a connected affine algebraic group defined over $k$. For any $k$-algebra $\mathfrak{A}$, $G(\mathfrak{A})$ stands for the set of $\mathfrak{A}$-rational points of $G$. Let $X^*(G)_k$ be the free $\mathbb{Z}$-module consisting of all $k$-rational characters of $G$. For each $g \in G(\mathbb{A})$, we define the homomorphism $\vartheta_G(g) : X^*(G)_k \to \mathbb{R}_{>0}$ by $\vartheta_G(g)(\chi) = |\chi(g)|_{\mathbb{A}}$ for $\chi \in X^*(G)_k$. Then $\vartheta_G$ is a homomorphism from $G(\mathbb{A})$ into $\text{Hom}_\mathbb{Z}(X^*(G)_k, \mathbb{R}_{>0})$. We write $G(\mathbb{A})^1$ for the kernel of $\vartheta_G$.

In the following, let $G$ be a connected isotropic reductive group defined over $k$. We fix a maximal $k$-split torus $S$ of $G$ and a minimal $k$-parabolic subgroup $P_0$ of $G$ containing $S$. Denote by $\Phi_k$ and $\Delta_k$ the relative root system of $G$ with respect to $S$ and the set of simple roots of $\Phi_k$ corresponding to $P_0$, respectively. Let $M_0$ be the centralizer of $S$ in $G$. Then $P_0$ has a Levi decomposition $P_0 = M_0U_0$, where $U_0$ is the unipotent radical of $P_0$. A $k$-parabolic subgroup of $G$ containing $P_0$ is called a standard $k$-parabolic subgroup of $G$. Every standard $k$-parabolic subgroup $R$ of $G$ has a unique Levi subgroup $M_R$ containing $M_0$. We denote by $U_R$ the unipotent radical of $R$ and by $Z_R$ the greatest central $k$-split torus in $M_R$. Throughout this paper, we fix a maximal compact subgroup $K = \prod_{\sigma \in p_\infty} K_\sigma \times \prod_{\sigma \in p_f} K_\sigma$ of $G(\mathbb{A})$ satisfying the following property: for every standard $k$-parabolic subgroup $R$ of $G$, $K \cap M_R(\mathbb{A})$ is a maximal compact subgroup of $M_R(\mathbb{A})$, and $M_R(\mathbb{A})$ possesses an Iwasawa decomposition $(M_R(\mathbb{A}) \cap U_0(\mathbb{A}))M_0(\mathbb{A})(K \cap M_R(\mathbb{A}))$.

Let $Q$ be a standard proper maximal $k$-parabolic subgroup of $G$. There is only one simple root $\alpha_0 \in \Delta_k$ such that the restriction of $\alpha_0$ to $Z_Q$ is nontrivial. Let $n_Q$ be the positive integer such that $n_Q^{-1}\alpha_0 | Z_Q$ is a $\mathbb{Z}$-basis of $X^*(Z_Q/Z_G)_k$. We write $\alpha_Q$ for $n_Q^{-1}\alpha_0 | Z_Q$ and $\widehat{\alpha}_Q$ for $\widehat{d}_Q n_Q^{-1}\alpha_0 | Z_Q$, where

$$\widehat{d}_Q = [X^*(Z_Q/Z_G)_k : X^*(M_Q/Z_G)_k].$$

Then $\widehat{\alpha}_Q$ is a $\mathbb{Z}$-basis of the submodule $X^*(M_Q/Z_G)_k$ of $X^*(Z_Q/Z_G)_k$. Define...
the map \( z_Q : G(\mathbb{A}) \to Z_G(\mathbb{A}) M_Q(\mathbb{A})^{-1} \setminus M_Q(\mathbb{A}) \) by \( z_Q(g) = Z_G(\mathbb{A}) M_Q(\mathbb{A})^{-1} m \) if \( g = umh \) with \( u \in U_Q(\mathbb{A}) \), \( m \in M_Q(\mathbb{A}) \) and \( h \in K \). This is well defined and left \( Z_G(\mathbb{A}) Q(\mathbb{A})^{-1} \)-invariant. Since \( Z_G(\mathbb{A})^{-1} = Z_G(\mathbb{A}) \cap G(\mathbb{A})^{-1} \subset M_Q(\mathbb{A})^{-1} \), \( z_Q \) gives rise to a map from \( Y_Q = Q(\mathbb{A})^{-1} \setminus G(\mathbb{A})^{-1} \) to \( M_Q(\mathbb{A})^{-1} \setminus (M_Q(\mathbb{A}) \cap G(\mathbb{A}))^{-1} \). Namely, we have the following commutative diagram, whose vertical arrows are natural maps:

\[
\begin{array}{ccc}
Y_Q & \xrightarrow{z_Q} & M_Q(\mathbb{A})^{-1} \setminus (M_Q(\mathbb{A}) \cap G(\mathbb{A}))^{-1} \\
\downarrow & & \downarrow \\
Z_G(\mathbb{A}) Q(\mathbb{A})^{-1} \setminus G(\mathbb{A}) & \xrightarrow{z_Q} & Z_G(\mathbb{A}) M_Q(\mathbb{A})^{-1} \setminus M_Q(\mathbb{A})
\end{array}
\]

We define the height function \( H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0} \) by \( H_Q(g) = |\hat{\alpha}_Q(z_Q(g))|_{\mathbb{A}}^{-1} \) for \( g \in G(\mathbb{A}) \). We notice that the restriction of \( H_Q \) to \( M_Q(\mathbb{A}) \) is a homomorphism from \( M_Q(\mathbb{A}) \) onto \( \mathbb{R}_{>0} \).

**Example 1.** Let \( G \) be a general linear group \( \text{GL}_n \) defined over the rational number field \( \mathbb{Q} \), \( P_0 \) the group of upper triangular matrices in \( G \) and \( S \) the group of diagonal matrices in \( G \). We fix an integer \( k \in \{1, \ldots, n-1\} \), and let

\[
Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \text{GL}_k(\mathbb{Q}), \ b \in M_{k,n-k}(\mathbb{Q}), \ d \in \text{GL}_{n-k}(\mathbb{Q}) \right\}.
\]

Then \( Q \) is a standard maximal \( \mathbb{Q} \)-parabolic subgroup of \( G \). The rational character \( \hat{\alpha}_Q \) and the height \( H_Q \) are given by

\[
\hat{\alpha}_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = (\det a)^{(n-k)/r} (\det d)^{-k/r}
\]

and

\[
H_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = |\det a|_{\mathbb{A}}^{-(n-k)/r} |\det d|_{\mathbb{A}}^{k/r},
\]

where \( r \) denotes the greatest common divisor of \( k \) and \( n-k \). The height \( H_Q \) has another expression. To explain this, let \( \mathbb{Q}^n \) be an \( n \)-dimensional column vector space over \( \mathbb{Q} \) with standard basis \( e_1, \ldots, e_n \). The maximal parabolic subgroup \( Q(\mathbb{Q}) \) stabilizes the subspace spanned by \( e_1, \ldots, e_k \). Let \( V_{n,k}(\mathbb{Q}) = \bigwedge^k \mathbb{Q}^n \) be the \( k \)-th exterior product of \( \mathbb{Q}^n \). We set \( V_{n,k}(\mathbb{A}) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A} \) and \( V_{n,k}(\mathbb{Q}_\sigma) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\sigma \) for \( \sigma \in \mathfrak{p}_\infty \cup \mathfrak{p}_f \). A \( \mathbb{Q} \)-basis of \( V_{n,k}(\mathbb{Q}) \) is formed by the elements \( e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \) with \( I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \ldots, n\} \). For a unique infinite place \( \infty \in \mathfrak{p}_\infty \), we define the local height \( H_\infty : V_{n,k}(\mathbb{Q}_\infty) \to \mathbb{R}_{>0} \) by

\[
H_\infty \left( \sum_I a_I e_I \right) = \left( \sum_I |a_I|_{\infty}^2 \right)^{1/2},
\]
where \( | \cdot |_{\infty} \) denotes the usual absolute value of \( \mathbb{Q}_{\infty} = \mathbb{R} \). For each finite prime \( p \in p_f \), we define the local height \( H_p : V_{n,k}(\mathbb{Q}) \to \mathbb{R}_{>0} \) by

\[
H_p \left( \sum_I a_I e_I \right) = \sup_I |a_I|_p,
\]

where \( | \cdot |_p \) denotes the \( p \)-adic absolute value of \( \mathbb{Q}_p \) normalized so that \( |p|_p = p^{-1} \). Then the global height \( H_{n,k} : V_{n,k}(\mathbb{Q}) \to \mathbb{R}_{>0} \) is defined to be a product of all local heights, that is, \( H_{n,k}(x) = \prod_{\sigma \in p_\infty \cup p_f} H_\sigma(x) \) for \( x \in V_{n,k}(\mathbb{Q}) \). This \( H_{n,k} \) is immediately extended to the subset \( GL(V_{n,k}(\mathbb{A}))V_{n,k}(\mathbb{Q}) \) of the adele space \( V_{n,k}(\mathbb{A}) \) by

\[
H_{n,k}(Ax) = \prod_{\sigma \in p_\infty \cup p_f} H_\sigma(A_\sigma x)
\]

for \( A = (A_\sigma) \in GL(V_{n,k}(\mathbb{A})) \) and \( x \in V_{n,k}(\mathbb{Q}) \). In particular, for \( g \in G(\mathbb{A}) = GL_n(\mathbb{A}) \), we can take the value \( H_{n,k}(ge_1 \wedge ge_2 \wedge \cdots \wedge ge_k) \). We choose a maximal compact subgroup \( K_\infty \) of \( G(\mathbb{Q}_\infty) \) as \( \{ g \in G(\mathbb{Q}_\infty) : |g|^{-1} = g \} \). Let

\[
K_f = \prod_{p \in p_f} GL_n(\mathbb{Z}_p) \quad \text{and} \quad K = K_\infty \times K_f.
\]

Then, by elementary computations, we have

\[
H_{n,k}(ge_1 \wedge ge_2 \wedge \cdots \wedge ge_k) = \left| \det a \right|_\mathbb{A} \quad \text{if} \quad g = h \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}
\]

with \( h \in K, \ a \in GL_k(\mathbb{A}), \ b \in M_{k,n-k}(\mathbb{A}) \) and \( d \in GL_{n-k}(\mathbb{A}) \). Therefore, if \( g \in G(\mathbb{A})^1 \), that is, \( \left| \det g \right|_\mathbb{A} = 1 \), then

\[
H_Q(g) = H_{n,k}(g^{-1}e_1 \wedge g^{-1}e_2 \wedge \cdots \wedge g^{-1}e_k)^{n/r}.
\]

2. Twisted height functions restricted to one parameter subgroups

Let \( N_G(S) \) be the normalizer of \( S \) in \( G \) and \( W_G = N_G(S)(k)/M_0(k) \) the Weyl group of \( G \) with respect to \( S \). For a simple root \( \alpha \in \Delta_k, \ s_\alpha \in W_G \) denotes the simple reflection corresponding to \( \alpha \). Then \( \{ s_\alpha \}_{\alpha \in \Delta_k} \) generates \( W_G \). We denote by \( W^O_G \) the subgroup of \( W_G \) generated by \( \{ s_\alpha \}_{\alpha \in \Delta_k \setminus \{ \alpha_0 \}} \). For each \( w \in W_G \), we use the same notation \( w \) for a representative of \( w \) in \( N_G(S)(k) \). The following cell decomposition of \( G(k) \) holds via Bruhat decomposition [Borel and Tits 1965, Proposition 4.10, Corollaire 5.20]:

\[
G(k) = \bigsqcup_{[w] \in W^O_G\backslash W_G/W^O_G} Q(k)wQ(k),
\]

where \([w]\) stands for the class \( W^O_GwW^O_G \) in \( W^O_G\backslash W_G/W^O_G \).


The Weyl group $W_G$ acts on $X^*(S)_k$ by $w \cdot \chi : t \mapsto \chi(w^{-1}tw)$ for $w \in W_G$ and $\chi \in X^*(S)_k$. We consider the restriction $\hat{\alpha}_Q|_S$ of the rational character $\hat{\alpha}_Q$ of $M_Q$ to $S$.

**Lemma 1.** The subgroup of $W_G$ fixing $\hat{\alpha}_Q|_S$ is equal to $W^Q_G$.

**Proof.** Put $W' = \{w \in W_G : w \cdot \hat{\alpha}_Q|_S = \hat{\alpha}_Q|_S\}$. Since a representative of $w \in W^Q_G$ is contained in $M_Q(k)$, we have $\hat{\alpha}_Q(w^{-1}tw) = \hat{\alpha}_Q(w)^{-1}\hat{\alpha}_Q(t)\hat{\alpha}_Q(w) = \hat{\alpha}_Q(t)$ for all $t \in S$. Hence $W^Q_G$ is contained in $W'$. By [Humphreys 1990, §1.12 Theorem (a) and (c)] , $W'$ is generated by a subset $W' \cap \{s_\alpha\}_{\alpha \in \Delta_k}$ of simple reflections. From $W^Q_G \subset W'$, it follows $\{s_\alpha\}_{\alpha \in \Delta_k \setminus \{\alpha_0\}} \subset W' \cap \{s_\alpha\}_{\alpha \in \Delta_k} \subset \{s_\alpha\}_{\alpha \in \Delta_k}$. Since $\hat{\alpha}_Q$ is nontrivial on $S/Z_G$, $W' \cap \{s_\alpha\}_{\alpha \in \Delta_k}$ must equal $\{s_\alpha\}_{\alpha \in \Delta_k \setminus \{\alpha_0\}}$. Therefore $W'$ coincides with $W^Q_G$. \hfill $\square$

Let $X^*_*(S)_k$ be the free $\mathbb{Z}$-module consisting of all $k$-rational cocharacters of $S$. A natural pairing

$$\langle \cdot, \cdot \rangle : X^*_*(S)_k \times X^*_*(S)_k \to \mathbb{Z}$$

defined as in [Borel 1991, §8.6] is a regular pairing over $\mathbb{Z}$.

**Lemma 2.** Let $w_1$ and $w_2$ be elements of $W_G$ such that $w_1^{-1}W^Q_G \neq w_2^{-1}W^Q_G$. Then there exist a cocharacter $\xi = \xi_{w_1w_2} \in X^*_*(S)_k$ such that

$$H_Q(w_1\xi(\lambda)w_1^{-1}) > H_Q(w_2\xi(\lambda)w_2^{-1})$$

holds for all $\lambda \in \mathbb{A}^\times_{>1}$, where $\mathbb{A}^\times_{>1}$ denotes the set of $\lambda \in \mathbb{A}^\times$ satisfying $|\lambda|_{\mathbb{A}} > 1$.

**Proof.** Since $w_1^{-1} \cdot \hat{\alpha}_Q|_S - w_2^{-1} \cdot \hat{\alpha}_Q|_S \neq 0$ by Lemma 1, there is a $\xi \in X^*_*(S)_k$ such that $\langle w_1^{-1} \cdot \hat{\alpha}_Q|_S - w_2^{-1} \cdot \hat{\alpha}_Q|_S, \xi \rangle < 0$. The value $\ell = \langle w_1^{-1} \cdot \hat{\alpha}_Q|_S - w_2^{-1} \cdot \hat{\alpha}_Q|_S, \xi \rangle$ is a negative integer. We have

$$\hat{\alpha}_Q(w_1\xi(\lambda)w_1^{-1}) \cdot \hat{\alpha}_Q(w_2\xi(\lambda)w_2^{-1})^{-1} = \lambda^\ell$$

for all $\lambda \in G_m$. Therefore,

$$H_Q(w_1\xi(\lambda)w_1^{-1})H_Q(w_2\xi(\lambda)w_2^{-1})^{-1} = |\lambda|_{\mathbb{A}}^{-\ell} > 1$$

holds for all $\lambda \in \mathbb{A}^\times_{>1}$. \hfill $\square$

3. **The Hermite function associated to $Q$ and minimal points**

We set $X_Q = Q(k) \setminus G(k)$, which is regarded as a subset of $Y_Q = Q(\mathbb{A})^1 \setminus G(\mathbb{A})^1$. Let $\pi_X : G(k) \to X_Q$ be the natural quotient map. The symbol $\tilde{e} = \pi_X(e) \in X_Q$ denotes the class of the unit element $e \in G(k)$. The Hermite function

$$m_Q : G(\mathbb{A})^1 \to \mathbb{R}_{>0}$$
is defined to be
\[ m_Q(g) = \min_{x \in X_Q} H_Q(xg). \]

By definition, \( m_Q \) is a positive valued continuous function on \( G(k) \backslash G(\mathbb{A})^1 / K \).

For each \( g \in G(\mathbb{A})^1 \), we put
\[ X_Q(g) = \{ x \in X_Q : m_Q(g) = H_Q(xg) \}, \]
which is a finite subset of \( X_Q \). Thus we can define the counting function \( n_Q(g) = \#X_Q(g) \).

**Lemma 3.** For any \( g \in G(\mathbb{A})^1 \), \( \gamma \in G(k) \) and \( h \in K \), one has \( X_Q(\gamma gh) = X_Q(g)\gamma^{-1} \). Especially, the counting function \( n_Q \) is left \( G(k) \)-invariant and right \( K \)-invariant.

The following lemma is proved by the same method as in [Watanabe 2012, Proof of Proposition 4.1].

**Lemma 4.** For \( g \in G(\mathbb{A})^1 \), there is a neighborhood \( \mathcal{U} \) of \( g \) in \( G(\mathbb{A})^1 \) such that \( X_Q(g') \subset X_Q(g) \) for all \( g' \in \mathcal{U} \).

**Example 2.** Let \( G \) be a general linear group \( GL_n \) defined over \( \mathbb{Q} \). We keep notations used in Example 1. In this case, we can express \( m_Q \) in terms of some minimum of positive definite symmetric matrices. Since \( GL_n / \mathbb{Q} \) is of class number one, \( G(\mathbb{A})^1 = \{ g \in GL_n(\mathbb{A}) : |\det g|_\mathbb{A} = 1 \} \) has the following decomposition:
\[ G(\mathbb{A})^1 = G(\mathbb{Q})(G(\mathbb{Q}_\infty)^1 \times K_f), \]
where \( G(\mathbb{Q}_\infty)^1 = \{ g \in GL_n(\mathbb{Q}_\infty) : \det g = \pm 1 \} \) and \( K_f = \prod_{p \in \mathcal{P}} GL_n(\mathbb{Z}_p) \). We fix \( g = \delta(g_\infty \times g_f) \in G(\mathbb{A})^1 \) with \( \delta \in G(\mathbb{Q}) \), \( g_\infty \in G(\mathbb{Q}_\infty)^1 \) and \( g_f \in K_f \). From the left \( G(\mathbb{Q}) \)-invariance and the right \( K \)-invariance of \( m_Q \), it follows that
\[ m_Q(g) = m_Q(g_\infty) = \min_{x \in X_Q} H_Q(xg_\infty) = \min_{\gamma \in G(\mathbb{Q})} H_Q(\gamma g_\infty). \]
Furthermore, since \( G(\mathbb{Q}) = Q(\mathbb{Q}) GL_n(\mathbb{Z}) \) and \( H_Q \) is left \( Q(\mathbb{Q}) \)-invariant, we have
\[ m_Q(g) = \min_{\gamma \in GL_n(\mathbb{Z})} H_Q(\gamma g_\infty). \]
An elementary proof of the decomposition \( G(\mathbb{Q}) = Q(\mathbb{Q}) GL_n(\mathbb{Z}) \) is found in [Shimura 1994, Theorem 3]. By Example 1,
\[
H_Q(\gamma g_\infty) = H_{n,k}(g_\infty^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_\infty^{-1} \gamma^{-1} e_k)^{n/r}
= H_\infty(g_\infty^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_\infty^{-1} \gamma^{-1} e_k)^{n/r} \prod_{p \in \mathcal{P}} H_p(\gamma^{-1} e_1 \wedge \cdots \gamma^{-1} e_k)^{n/r}
= H_\infty(g_\infty^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_\infty^{-1} \gamma^{-1} e_k)^{n/r}.
\]
Here we notice that $H_p(\gamma^{-1}e_1 \wedge \cdots \wedge \gamma^{-1}e_k) = 1$ for all $p \in p_f$ and $\gamma \in \text{GL}_n(\mathbb{Z})$. For a given $\gamma \in \text{GL}_n(\mathbb{Z})$, $X_\gamma$ stands for the $n$ by $k$ matrix consisting of the first $k$ columns of $\gamma$. Binet’s formula (see [Bombieri and Gubler 2006, Proposition 2.8.8]) yields

$$H_\infty(g_\infty^{-1}\gamma^{-1}e_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}e_k) = \det(t^TX_\gamma^{-1}t g_\infty^{-1}g_\infty^{-1}X_\gamma^{-1})^{1/2}.$$ 

As a consequence, we obtain

$$m_Q(g) = \min_{X \in \mathbb{M}_{n,k}(\mathbb{Z})^*} \det(t^X t g_\infty^{-1}g_\infty^{-1}X)^{n/2r},$$

where $\mathbb{M}_{n,k}(\mathbb{Z})^*$ denotes the set of $X_\gamma$ for all $\gamma \in \text{GL}_n(\mathbb{Z})$. In the case of $k = 1$, $\mathbb{M}_{n,1}(\mathbb{Z})^*$ is just the set of primitive vectors of the lattice $\mathbb{Z}^n$, and hence $m_Q(g)$ coincides with the $n/2$ power of the arithmetical minimum of the positive definite symmetric matrix $t g_\infty^{-1}g_\infty^{-1}$.

4. The Ryshkov domain of $G$ associated to $Q$

We define the Ryshkov domain $R = R(m_Q)$ of $m_Q$ by

$$R = R(m_Q) = \{g \in G(\mathbb{A})^1 : m_Q(g)/H_Q(g) \geq 1\}.$$ 

Since $m_Q(g) \leq H_Q(g)$ holds for all $g \in G(\mathbb{A})^1$, we have

$$R = \{g \in G(\mathbb{A})^1 : m_Q(g) = H_Q(g)\} = \{g \in G(\mathbb{A})^1 : \tilde{e} \in X_Q(g)\}.$$ 

Since both $H_Q$ and $m_Q$ are continuous, $R$ is a closed subset in $G(\mathbb{A})^1$.

**Lemma 5.** One has $Q(k)R K = R$ and $G(\mathbb{A})^1 = G(k)R$.

**Proof.** The first assertion is obvious by the definition of $H_Q$. To prove the second assertion, we choose a minimal point $x \in X_Q(g)$ for a given $g \in G(\mathbb{A})^1$. There is a $\gamma \in G(k)$ such that $x = \pi_X(\gamma)$. Then $H_Q(xg) = H_Q(\gamma g) = m_Q(g) = m_Q(\gamma g)$ since $m_Q$ is left $G(k)$-invariant. Therefore, $\gamma g \in R$. \qed

**Lemma 6.** Let $C$ be an arbitrary subset of $G(\mathbb{A})^1$, and let $g \in G(\mathbb{A})^1$ and $\gamma \in G(k)$.

1. $\gamma g \in R$ if and only if $\pi_X(\gamma) \in X_Q(g)$.
2. $X_Q(g) = \pi_X(\{\gamma \in G(k) : \gamma g \in R\})$.
3. $\gamma C \subset R$ if and only if $\pi_X(\gamma) \in \bigcap_{g \in C} X_Q(g)$.
4. $\bigcap_{g \in R} X_Q(g) = \{\tilde{e}\}$.
5. $\gamma R \subset R$ if and only if $\gamma \in Q(k)$. 

Proof. By definition, \( \gamma g \in R \) if and only if \( m_Q(\gamma g) = H_Q(\gamma g) \). This is equivalent to \( \pi_X(\gamma) \in X_Q(g) \) because \( m_Q(\gamma g) = m_Q(g) \). Both (2) and (3) follow from (1). For a point \( x = \pi_X(\gamma) \in \bigcap_{g \in R} X_Q(g) \), we have \( \gamma Q(k)R \subseteq R \); in other words, \( x Q(k) \subseteq \bigcap_{g \in R} X_Q(g) \). Since \( x Q(k) \) is an infinite set for \( x \neq \tilde{e} \) by Bruhat decomposition, we must have \( x = \tilde{e} \). This shows (4). Item (5) follows from (3) and (4).

\[ 2 \]

Lemma 7. Let \( g_0 \in R \) be an element such that \( n_Q(g_0) > 1 \) and \( x_0 \) an arbitrary element in \( X_Q(g_0) \). Then, any neighborhood \( \mathfrak{U} \) of \( g_0 \) in \( G(\mathbb{A}) \) contains a point \( g \) such that \( X_Q(g) \subset X_Q(g_0) \) and \( x_0 \not\in X_Q(g) \).

Proof. We may assume \( \mathfrak{U} \) satisfies \( X_Q(g) \subset X_Q(g_0) \) for all \( g \in \mathfrak{U} \) by Lemma 4. Since \( n_Q(g_0) > 1 \), there is an \( x \in X_Q(g_0) \) such that \( x \neq \tilde{e} \). This \( x \) is of the form \( \pi_X(w \gamma) \) with \( w \in W_G \setminus W_G^Q \) and \( \gamma \in Q(k) \). By Lemma 2, there is a cocharacter \( \xi = \xi_{w,e} \in X_*(S) \) such that \( H_Q(w \xi(\lambda)w^{-1}) > H_Q(\xi(\lambda)) \) holds for all \( \lambda \in \mathbb{A}_x^{>1} \). Let \( \lambda \in \mathbb{A}_x^{>1} \) be an element sufficiently close to 1 so that \( g_\lambda = \gamma^{-1}\xi(\lambda)g_0 \) is contained in \( \mathfrak{U} \). We have

\[
H_Q(g_\lambda) = H_Q(\xi(\lambda)g_0) = H_Q(\xi(\lambda))H_Q(\gamma g_0)
= H_Q(\xi(\lambda))H_Q(g_0) = H_Q(\xi(\lambda))m_Q(g_0)
\]

and

\[
H_Q(xg_\lambda) = H_Q(w \xi(\lambda)g_0) = H_Q(w \xi(\lambda)w^{-1})H_Q(w \gamma g_0)
= H_Q(w \xi(\lambda)w^{-1})m_Q(g_0).
\]

If \( x_0 = \tilde{e} \), then we choose \( \lambda \) sufficiently close to 1 satisfying \( \lambda^{-1} \in \mathbb{A}_x^{>1} \). Since \( X_Q(g_\lambda) \subset X_Q(g_0) \) and \( m_Q(g_\lambda) \leq H_Q(xg_\lambda) < H_Q(g_\lambda) \), \( X_Q(g_\lambda) \) does not contain \( \tilde{e} \). If \( x_0 \neq \tilde{e} \), then we choose \( x \) as \( x_0 \) and \( \lambda \in \mathbb{A}_x^{>1} \) sufficiently close to 1. Since \( m_Q(g_\lambda) \leq H_Q(g_\lambda) < H_Q(x_0g_\lambda) \), \( X_Q(g_\lambda) \) does not contain \( x_0 \).

\[ 3 \]

Lemma 8. \( \min_{g \in G(\mathbb{A})^1} n_Q(g) = \min_{g \in R} n_Q(g) = 1 \).

Proof. From Lemma 5 and the \( G(k) \)-invariance of \( n_Q \), it follows that

\[
\min_{g \in G(\mathbb{A})^1} n_Q(g) = \min_{g \in R} n_Q(g).
\]

If \( g_0 \in R \) satisfies \( \min_{g \in R} n_Q(g) = n_Q(g_0) > 1 \), then by Lemmas 5 and 7, there exist a point \( g_1 \in G(\mathbb{A})^1 \) and \( \gamma_1 \in G(k) \) such that \( n_Q(\gamma_1 g_1) = n_Q(g_1) < n_Q(g_0) \) and \( \gamma_1 g_1 \in R \). This is a contradiction.

We define the subset \( R_1 \) of \( R \) by

\[
R_1 = \{ g \in R : n_Q(g) = 1 \} = \{ g \in G(\mathbb{A})^1 : X_Q(g) = \{ \tilde{e} \} \}.
\]

Lemma 9. \( R_1 \) coincides with the interior \( R^o \) of \( R \) in \( G(\mathbb{A})^1 \).
Proof. For $g \in R_1$, we choose a neighborhood $\mathcal{U}$ of $g$ in $G(\mathbb{A})^1$ as in Lemma 4. Then $\mathcal{U} \subset R_1$. Therefore, $R_1$ is open and is contained in $R^\circ$. If there exists an element $g_0 \in R^\circ$ such that $n_Q(g_0) > 1$, then, by Lemma 7, $R^\circ$ contains an element $g$ satisfying $\hat{e} \not\in X_Q(g)$. This contradicts $g \in R$. \hfill \Box

It is obvious that $G(k)R_1 = \{ g \in G(\mathbb{A})^1 : n_Q(g) = 1 \}$.

Lemma 10. $G(k)R_1$ is open and dense in $G(\mathbb{A})^1$.

Proof. Since $R_1$ is open in $G(\mathbb{A})^1$, so is $G(k)R_1$. We assume $G(\mathbb{A})^1 \setminus G(k)R_1$ has an interior point $g_0$. Let $\mathcal{U}$ be a neighborhood of $g_0$ in $G(\mathbb{A})^1$ so that $\mathcal{U} \cap G(k)R_1 = \emptyset$. By Lemma 5, we can take $\gamma_0 \in G(k)$ such that $\gamma_0 g_0 \in R$. Since $n_Q(\gamma_0 g_0) = n_Q(g_0) > 1$, by Lemmas 5 and 7, there exist $g_1 \in \gamma_0 \mathcal{U}$ and $\gamma_1 \in G(k)$ such that $n_Q(g_1) < n_Q(g_0)$ and $\gamma_1 g_1 \in R$. If $n_Q(g_1) > 1$, then there exist $g_2 \in \gamma_1 \gamma_0 \mathcal{U}$ and $\gamma_2 \in G(k)$ such that $n_Q(g_2) < n_Q(g_1)$ and $\gamma_2 g_2 \in R$. This process terminates after finitely many iterations. At the last step, we obtain an element $g_\ell \in \gamma_{\ell-1} \cdots \gamma_0 \mathcal{U}$ such that $n_Q(g_\ell) = 1$. Then $(\gamma_{\ell-1} \cdots \gamma_0)^{-1} g_\ell$ is contained in $\mathcal{U} \cap G(k)R_1$. This contradicts $\mathcal{U} \cap G(k)R_1 = \emptyset$. Therefore, $G(\mathbb{A})^1 \setminus G(k)R_1$ is nowhere dense in $G(\mathbb{A})^1$. \hfill \Box

Lemma 11. For $\gamma \in G(k)$, $R_1 \cap \gamma R \neq \emptyset$ if and only if $\gamma \in Q(k)$.

Proof. If $R_1 \cap \gamma R$ has an element $g$, then $\pi_X(\gamma^{-1}) \in X_Q(g) = \{ \hat{e} \}$ by Lemma 6. \hfill \Box

Lemma 12. Let $R_1^-$ be the closure of $R_1$. Then we have the following subdivision of $G(\mathbb{A})^1$:

$$G(\mathbb{A})^1 = \bigcup_{\gamma Q(k) \in G(k)/Q(k)} \gamma R_1^-.$$

Proof. We fix an arbitrary $g \in G(\mathbb{A})^1$. By Lemma 10, there exists a sequence \{gn\} \subset G(k)R_1 such that $\lim_{n \to \infty} g_n = g$. We take a neighborhood $\mathcal{U}$ of $g$ as in Lemma 4 and may assume that \{gn\} \subset \mathcal{U}. Since $g_n \in G(k)R_1$, $X_Q(g_n)$ consists of a single element $\pi_X(\gamma_n)$, where $\gamma_n \in G(k)$. From $g_n \in \mathcal{U}$, it follows that $\pi_X(\gamma_n) \in X_Q(g)$ for all $n$. Since $X_Q(g)$ is a finite set, we can take a subsequence \{gn_j\} such that $\pi_X(\gamma_{n_j}) = \pi_X(\gamma) \in X_Q(g)$ for all $n_j$. Then \{gn_j\} \subset $\gamma^{-1}R_1$, and $g$ is contained in the closure of $\gamma^{-1}R_1$. \hfill \Box

For $g \in G(\mathbb{A})^1$, we put

$$S_Q(g) = \pi_X(\{ \gamma \in G(k) : \gamma g \in R_1^- \}).$$

By Lemmas 6 and 12, $S_Q(g)$ is a nonempty subset of $X_Q(g)$.

Lemma 13. For $g_0 \in G(\mathbb{A})^1$, there is a neighborhood $\mathcal{U}$ of $g_0$ in $G(\mathbb{A})^1$ such that $S_Q(g) \subset S_Q(g_0)$ for all $g \in \mathcal{U}$.
**Proof.** Let \( \mathcal{U} \) be a neighborhood of \( g_0 \) such that \( X_Q(g) \subset X_Q(g_0) \) for all \( g \in \mathcal{U} \). Since \( g_0 \not\in \gamma^{-1}R_1^{-} \) for any \( \pi_{X}(\gamma) \in X_Q(g_0) \setminus S_Q(g_0) \), we can take a sufficiently small \( \mathcal{U} \) so that \( \mathcal{U} \cap \gamma^{-1}R_1^{-} = \emptyset \) for all \( \pi_{X}(\gamma) \in X_Q(g_0) \setminus S_Q(g_0) \). Then, for any \( g \in \mathcal{U} \), \( S_Q(g) \cap X_Q(g_0) \setminus S_Q(g_0) \) is empty; that is, \( S_Q(g) \subset S_Q(g_0) \). \( \square \)

**Remark.** We do not know whether \( R_1^{-} = R \) holds or not in general. If it does, then \( S_Q(g) = X_Q(g) \) holds for all \( g \).

5. A fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \)

**Definition.** Let \( T \) be a locally compact Hausdorff space and \( \Gamma \) be a discrete group acting on \( T \) from the left. Assume that the action of \( \Gamma \) on \( T \) is properly discontinuous. An open subset \( \Omega \) of \( T \) is called an open fundamental domain of \( T \) with respect to \( \Gamma \) if \( \Omega \) satisfies the following conditions:

1. \( T = \Gamma \Omega^- \), where \( \Omega^- \) stands for the closure of \( \Omega \) in \( T \), and
2. \( \Omega \cap \gamma \Omega^- = \emptyset \) if \( \gamma \in \Gamma \setminus \{e\} \).

A subset \( F \) of \( T \) is called a fundamental domain of \( T \) with respect to \( \Gamma \) if there is an open fundamental domain \( \Omega \) as above such that \( \Omega \subset F \subset \Omega^- \).

By Baer and Levi’s theorem [1931] (see also [van der Waerden 1935, §10]), an open fundamental domain of \( T \) with respect to \( \Gamma \) exists if the set of points stabilized by some nontrivial element of \( \Gamma \) is discrete in \( T \). Thus there exists an open fundamental domain \( \Omega_Q \) of \( R_1^{-} \) with respect to \( Q(k) \).

For a given subset \( A \) of \( R_1^{-} \), \( A^\circ \) and \( A^- \) denote the interior and the closure of \( A \) in \( G(\mathbb{A})^1 \), respectively. Since \( R_1^{-} \) is closed in \( G(\mathbb{A})^1 \), the closure of \( A \) in \( R_1^{-} \) coincides with \( A^- \).

**Lemma 14.** Let \( \Omega_Q \) be an open fundamental domain of \( R_1^{-} \) with respect to \( Q(k) \). Then one has \( \Omega_Q^\circ = \Omega_Q \cap R_1 \) and \( \Omega_Q^- = (\Omega_Q \cap R_1)^- \).

**Proof.** Since \( \Omega_Q \) is an open set in \( R_1^{-} \) with respect to the relative topology, there is an open set \( \mathcal{U} \) in \( G(\mathbb{A})^1 \) such that \( \Omega_Q = R_1^{-} \cap \mathcal{U} \). Therefore, \( \Omega_Q \cap R_1 = \mathcal{U} \cap R_1 \) is open in \( G(\mathbb{A})^1 \), and hence \( \Omega_Q^\circ = \Omega_Q \cap R_1 \). Since \( R_1 \) is dense in \( R_1^{-} \) and \( \Omega_Q \) is relatively open in \( R_1^{-} \), the closure of \( \Omega_Q \cap R_1 \) in \( R_1^{-} \) contains \( \Omega_Q \), that is, \( \Omega_Q \subset (\Omega_Q \cap R_1)^- \). Hence \( \Omega_Q^- = (\Omega_Q \cap R_1)^- \). \( \square \)

**Theorem 15.** Let \( \Omega_Q \) be an open fundamental domain of \( R_1^{-} \) with respect to \( Q(k) \). Then \( \Omega_Q^\circ \) is an open fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \).

**Proof.** From \( R_1^{-} = Q(k)\Omega_Q^- \) and Lemma 12, it follows \( G(\mathbb{A})^1 = G(k)\Omega_Q^- \). For \( \gamma \in G(k) \), we assume \( \Omega_Q^\circ \cap \gamma \Omega_Q^- \not\subset \emptyset \). By Lemma 11, \( \gamma \) is contained in \( Q(k) \). Since \( \Omega_Q \) is an open fundamental domain of \( R_1^{-} \) with respect to \( Q(k) \), \( \gamma \) must be equal to \( e \). \( \square \)

For a given subset \( A \) of \( G(\mathbb{A})^1 \), we denote by \( \partial A \) the boundary of \( A \).
Lemma 16. If $g_0 \in R_1$ attains a local maximum of $m_Q$, then $g_0$ is in $\partial R_1$.

Proof. Suppose $g_0 \in R_1$. Since $R_1$ is open, $zg_0$ is contained in $R_1$ if $z \in Z_Q(\mathbb{A})$ is sufficiently close to $e$. Then

$$m_Q(zg_0) = H_Q(zg_0) = H_Q(z)H_Q(g_0) = H_Q(z)m_Q(g_0).$$

Since $H_Q(z)$ can vary on the interval $(1 - \epsilon, 1 + \epsilon)$ for a sufficiently small $\epsilon > 0$, $m_Q(g_0)$ is not a local maximum of $m_Q$. \hfill \Box

Since $(\Omega_Q^\partial)^\circ = \Omega_Q^\partial \subset R_1$, the following theorem immediately follows from Lemma 16.

Theorem 17. Let $\Omega_Q$ be the same as in Theorem 15. If $g_0 \in \Omega_Q^\partial$ attains a local maximum of $m_Q$, then $g_0$ is in $\partial \Omega_Q^\partial \cap \partial R_1$.

Remark. A point $g_0 \in G(\mathbb{A})^1$ is said to be extreme if $g_0$ attains a local maximum of $m_Q$. By Theorem 17, any extreme point is contained in $G(k)(\partial \Omega_Q^\partial \cap \partial R_1^\partial)$. A candidate of the notion analogous to perfect quadratic forms is the following: a point $g \in G(\mathbb{A})^1$ is said to be $Q$-perfect if there is a neighborhood $U$ of $g$ such that

$$U \cap \bigcap_{\pi(x) \in S_Q(g)} \delta^{-1}R_1^\partial = \{g\}.$$

6. The case when $G$ is of class number one

We put $K_f = \prod_{\sigma \in p_f} K_{\sigma}$, $G_{\mathbb{A}, \infty} = G(k_\infty) \times K_f$, $G_{\mathbb{A}, \infty}^1 = G_{\mathbb{A}, \infty} \cap G(\mathbb{A})^1$ and $G_o = G(k) \cap G_{\mathbb{A}, \infty}$. By identifying $G(k_\infty)$ with the subgroup

$$\{(g_\sigma) \in G(\mathbb{A}) : g_\sigma = e \text{ for all } \sigma \in p_f\}$$

of $G(\mathbb{A})$, we put $G(k_\infty)^1 = G(k_\infty) \cap G(\mathbb{A})^1$. The number $n_k(G)$ of double cosets in $G(\mathbb{A})$ modulo $G(k)$ and $G_{\mathbb{A}, \infty}$ is called the class number of $G$. For example, $n_k(GL_n)$ is equal to the class number of $k$. If $G$ is almost $k$-simple, $k$-isotropic and simply connected, then $n_k(G) = 1$ by the strong approximation theorem. In this section, we assume that $n_k(G) = 1$. Then $G(\mathbb{A})^1 = G(k)G_{\mathbb{A}, \infty}^1$. Let $h_Q$ be the number of double cosets of $G(k)$ modulo $Q(k)$ and $G_o$. By [Borel 1963, Proposition 7.5], $h_Q$ is equal to the class number of $M_Q$. Let $\{\xi_1 = e, \xi_2, \ldots, \xi_{h_Q}\}$ be a complete system of representatives of $Q(k)\backslash G(k)/G_o$. For each $\xi_i$, we define

$$R_{\xi_i, \infty} = \{g_\infty \in G(k_\infty)^1 : m_Q(g_\infty) = H_Q(\xi_ig_\infty)\}.$$

Since $G(k)$ is a disjoint union of $Q(k)\xi_iG_o$ for $i = 1, \ldots, h_Q$, $m_Q(g_\infty)$ equals

$$\min_{1 \leq i \leq h_Q} \min_{g_\infty \in G_o} H_Q(\xi_ig_\infty).$$
Lemma 18. \[ R = \bigsqcup_{i=1}^{h_Q} Q(k)\xi_i (R_{\xi_i,\infty} \times K_f). \]

Proof. For each \( i \), \( Q(k)\xi_i (R_{\xi_i,\infty} \times K_f) \subset R \) is trivial. Since

\[ G(\mathbb{A})^1 = \bigsqcup_{i=1}^{h_Q} Q(k)\xi_i G_{\mathbb{A},\infty}^1 \]

by [Borel 1963, §7], a given \( g \in R \) is represented as \( g = \gamma \xi_i (g_{\infty} \times g_f) \) for some \( i, \gamma \in Q(k) \) and \( g_{\infty} \times g_f \in G_{\mathbb{A},\infty}^1 \). Then \( m_Q(g) = H_Q(g) \) implies \( m_Q(g_{\infty}) = H_Q(\xi_i g_{\infty}) \). Therefore, \( g_{\infty} \in R_{\xi_i,\infty} \).

We write \( Q_i \) for the conjugate \( \xi_i^{-1} Q \xi_i \) of \( Q \). This \( Q_i \) is a maximal \( k \)-parabolic subgroup of \( G \). We put \( Q_{i,0} = Q_i(k) \cap G_{\mathbb{A},\infty} \).

Lemma 19. If \( g(R_{\xi_i,\infty} \times K_f) \cap (R_{\xi_i,\infty} \times K_f) \) is nonempty for \( g \in Q_i(k) \), then \( g \in Q_{i,0} \).

Proof. If there is an \( h \in R_{\xi_i,\infty} \times K_f \) such that \( gh \in R_{\xi_i,\infty} \times K_f \), then \( g \in (R_{\xi_i,\infty} \times K_f) h^{-1} \subset G_{\mathbb{A},\infty} \).

It is easy to prove that the group \( Q_{i,0} \) stabilizes \( R_{\xi_i,\infty} \times K_f \) by left multiplication.

We fix a complete system \( \{ \gamma_{ij} \}_j \) of representatives of \( Q_i(k)/Q_{i,0} \). It follows from Lemma 19 that \( \gamma_{ij}(R_{\xi_i,\infty} \times K_f) \cap \gamma_{ik}(R_{\xi_i,\infty} \times K_f) = \emptyset \) if \( j \neq k \). Therefore, we obtain the following subdivision of \( R \):

\[ R = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i \gamma_{ij} (R_{\xi_i,\infty} \times K_f). \]

Let \( R_{\xi_i,\infty}^0 \) be the interior of \( R_{\xi_i,\infty} \) and \( R_{\xi_i,\infty}^* \) the closure of \( R_{\xi_i,\infty}^0 \) in \( G(k_{\infty})^1 \). Since the union of (1) is disjoint, it is obvious that

\[ R_1^- = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i \gamma_{ij} (R_{\xi_i,\infty}^* \times K_f). \]

Proposition 20. Let \( \Omega_{i,\infty} \) be an open fundamental domain of \( R_{\xi_i,\infty}^* \) with respect to \( Q_{i,0} \) for \( i = 1, \ldots, h_Q \). Then the set

\[ \Omega = \bigsqcup_{i=1}^{h_Q} \xi_i (\Omega_{i,\infty} \times K_f) \]

gives an open fundamental domain of \( R_1^- \) with respect to \( Q(k) \).
Proof. Let \( \Omega_{i,\infty}^- \) denote the closure of \( \Omega_{i,\infty} \) in \( G(k_\infty)^1 \). For \( g \in Q(k) \), we assume \( \Omega \cap g \Omega^- \neq \emptyset \). Then, for some \( i, j \),

\[
\xi_i(\Omega_{i,\infty} \times K_f) \cap g \xi_j(\Omega_{j,\infty}^- \times K_f) \neq \emptyset.
\]

There exist \( \gamma_{jk} \) and \( \delta \in Q_{j,o} \) such that \( \xi_j^{-1} g \xi_j = \gamma_{jk} \delta \). Then (3) is the same as

\[
\xi_i(\Omega_{i,\infty} \times K_f) \cap \xi_j \gamma_{jk}(\delta \Omega_{j,\infty}^- \times K_f) \neq \emptyset.
\]

By (1), we have \( i = j \), \( \gamma_{jk} = e \) and \( \Omega_{j,\infty} \cap \delta \Omega_{j,\infty}^- \neq \emptyset \). Since \( \Omega_{j,\infty} \) is an open fundamental domain of \( R^*_j,\infty \) with respect to \( Q_{j,o}, \delta \) must be equal to \( e \). Therefore, \( \Omega \cap g \Omega^- \neq \emptyset \) implies \( g = e \). Finally, \( Q(k) \Omega^- = R_1^- \) follows from (2) and \( Q_{i,o} \Omega_{i,\infty} = R^*_i,\infty \).

By Theorem 17, we obtain the following.

Corollary 21. If \( g_0 \in \Omega^- \) attains a local maximum of \( m_Q \), then \( g_0 \) is contained in the set

\[
\bigcup_{i=1}^{h_Q} \xi_i((\partial \Omega_{i,\infty}^- \cap \partial R^*_i,\infty) \times K_f).
\]

We consider the infinite part \( \Omega_\infty \) of \( \Omega \) given in Proposition 20, that is,

\[
\Omega_\infty = \bigcup_{i=1}^{h_Q} \xi_i \Omega_{i,\infty}^-.
\]

Let \( \Omega_\infty^\circ \) and \( \Omega_\infty^- \) be the interior and the closure of \( \Omega_\infty \) in \( G(k_\infty)^1 \), respectively. The projection from \( G(\mathbb{A})^1 = G(k) \mathbb{A}_{k,\infty} \) to the infinite component \( G(k_\infty)^1 \) gives an isomorphism \( G(k) \mathbb{A}_1^1/K_f \cong G_\infty \mathbb{A}(k_\infty)^1 \). Since \( \Omega \) is a fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \) by Theorem 15, we have \( G_\infty \Omega_\infty = G(k_\infty)^1 \).

Corollary 22. If \( h_Q = 1 \), then \( \Omega_\infty \) is a fundamental domain of \( G(k_\infty)^1 \) with respect to \( G_\infty \).

Proof. Since \( \Omega_\infty = \Omega_{1,\infty} \) is a relatively open set in \( R^*_e,\infty \), we have \( \Omega_\infty^\circ = \Omega_\infty \cap R^*_e,\infty \). Thus the closure of \( \Omega_\infty^\circ \) coincides with \( \Omega_\infty^- \). If \( \Omega_\infty^- \cap g \Omega_\infty^- \neq \emptyset \) for \( g \in G_\infty \), then \( (\Omega_\infty^\circ \times K_f) \cap g(\Omega_\infty^- \times K_f) \neq \emptyset \) because \( gK_f = K_f \). This implies \( g = e \) since \( \Omega_\infty^\circ \times K_f \) is an open fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(k) \).

7. Examples

Example 3. Let \( G \) be a general linear group \( GL_n \) defined over \( \mathbb{Q} \). We continue an illustration given in Examples 1 and 2. We fix an integer \( k \in \{1, \ldots, n-1\} \), and
We define the closed subset 

\[ Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \text{GL}_k(\mathbb{Q}), \ b \in \text{M}_{k,n-k}(\mathbb{Q}), \ d \in \text{GL}_{n-k}(\mathbb{Q}) \right\}. \]

Since \( h_Q = 1 \), we have \( \xi_1 = e \) and \( Q_1 = Q \).

Let \( P^1_n \) be the cone of positive definite \( n \) by \( n \) real symmetric matrices, and let \( P_n \) be the intersection of \( P^1_n \) and \( \text{SL}_n(\mathbb{R}) \). The group \( G(\mathbb{Q}_\infty) = \text{GL}_n(\mathbb{R}) \) acts on \( P_n \) from the right by \( (A, g) \mapsto A[g] = t^g Ag \) for \( (A, g) \in P_n \times G(\mathbb{Q}_\infty) \). The maximal compact subgroup \( K_\infty \) of \( G(\mathbb{Q}_\infty) \), defined as in Example 2, stabilizes the identity matrix \( I_n \in P_n \). The map \( \pi : g \mapsto t^g g^{-1} \) from \( G(\mathbb{Q}_\infty) \) onto \( P_n \) gives an isomorphism between \( G(\mathbb{Q}_\infty)/K_\infty \) and \( P_n \). Since 

\[ G(\mathbb{Q}_\infty)^1 = \{ g \in G(\mathbb{Q}_\infty) : \det g = \pm 1 \}, \]

we have \( G(\mathbb{Q}_\infty)^1/K_\infty \cong \pi(G(\mathbb{Q}_\infty)^1) = P^1_n \). An element \( A \in P_n \) is written as

\[ A = \begin{pmatrix} I_k & 0 \\ t^u & I_{n-k} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} I_k & u \\ 0 & I_{n-k} \end{pmatrix}, \]

where \( v \in P_k, \ w \in P_{n-k} \) and \( u \in \text{M}_{k,n-k}(\mathbb{R}) \). We write \( uA, \ A^k \) and \( A_{[n-k]} \) for \( u, \ v \) and \( w \), respectively.

By definition, \( G_Z = G(\mathbb{Q}) \cap G_{\mathbb{A},\infty} \) and \( Q_Z = Q(\mathbb{Q}) \cap G_{\mathbb{A},\infty} \) are just the groups \( \text{GL}_n(\mathbb{Z}) \) and \( Q(\mathbb{Q}) \cap \text{GL}_n(\mathbb{Z}) \) of unimodular integral matrices in \( G(\mathbb{Q}) \) and \( Q(\mathbb{Q}) \), respectively. As in Example 2, \( X_\gamma \) stands for the \( n \) by \( k \) matrix consisting of the first \( k \)-columns of \( \gamma \in G_Z \), and \( M_{n,k}(\mathbb{Z})^* \) stands for the set of \( X_\gamma \) for all \( \gamma \in G_Z \). We define the closed subset \( F_{n,k} \) of \( P_n \) as follows:

\[ F_{n,k} = \left\{ A \in P_n : \det A^k \leq \det(t^{XAX}) \text{ for all } X \in M_{n,k}(\mathbb{Z})^* \right\}. \]

In Example 2, we showed

\[ H_Q(\gamma g) = \det(t^{X_{\gamma^{-1}} \pi(g) X_{\gamma^{-1}}})^{n/2r} \]

for any \( \gamma \in G_Z \) and \( g \in G(\mathbb{Q}_\infty)^1 \). Since \( H_Q(g) = (\det \pi(g)^{[k]})^{n/2r} \), we obtain

\[ R_{e,\infty}/K_\infty \cong \pi(R_{e,\infty}) = F_{n,k} \cap \text{SL}_n(\mathbb{R}). \]

Therefore, \( Q_Z \setminus R_{e,\infty}/K_\infty \) is isomorphic to \( (F_{n,k} \cap \text{SL}_n(\mathbb{R}))/Q_Z \). If \( \gamma \in Q_Z \) is of the form

\[ \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \]

with \( a \in \text{GL}_k(\mathbb{Z}), \ d \in \text{GL}_{n-k}(\mathbb{Z}) \) and \( b \in \text{M}_{k,n-k}(\mathbb{Z}) \), then components of \( t^{\gamma} A \gamma \) for \( A \in P_n \) are given by

\[ u_{\gamma} A_{\gamma} = a^{-1}(u_A d + b), \quad (t^{\gamma} A \gamma)^{[k]} = t^{a A^{[k]}} a, \quad (t^{\gamma} A \gamma)_{[n-k]} = t^{d A_{[n-k]} d}. \]
Let $\mathcal{D}$ and $\mathcal{E}$ be arbitrary fundamental domains for the quotients $P_k/\text{GL}_k(\mathbb{Z})$ and $P_{n-k}/\text{GL}_{n-k}(\mathbb{Z})$, respectively. We define the subset $F_{n,k}(\mathcal{D}, \mathcal{E})$ of $F_{n,k}$ as

$$F_{n,k}(\mathcal{D}, \mathcal{E}) = \{ A \in F_{n,k} : A[k] \in \mathcal{D}, A[n-k] \in \mathcal{E},$$

$$u_A = (u_{ij}) = (-\frac{1}{2} \leq u_{ij} \leq \frac{1}{2} \text{ for all } i, j, \text{ and } 0 \leq u_{11}) \}.$$ 

Since $F_{n,k}(\mathcal{D}, \mathcal{E})$ is a fundamental domain of $F_{n,k}$ with respect to $Q_\mathbb{Z}$, the inverse image $\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R}))$ of $F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R})$ gives a fundamental domain of $R_{e,\infty}$ with respect to $Q_\mathbb{Z}$. As a consequence of Theorem 15 and Proposition 20, the set

$$\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R})) \times K_f$$

gives a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathbb{Q})$. Moreover, from Corollary 22, it follows that $F_{n,k}(\mathcal{D}, \mathcal{E})$ is a fundamental domain of $P_n$ with respect to $\text{GL}_n(\mathbb{Z})$.

In the case of $k = 1$, this gives an inductive construction of a fundamental domain $\Omega_n$ of $P_n$ with respect to $\text{GL}_n(\mathbb{Z})$ as follows. First, put $\Omega_2 = F_{2,1}(P_1, P_1)$. By definition, $\Omega_2$ is Minkowski’s fundamental domain of $P_2$. Then we define inductively $\Omega_3 = F_{3,1}(P_1, \Omega_2), \ldots, \Omega_n = F_{n,1}(P_1, \Omega_{n-1})$. The domain $\Omega_n$ coincides with Grenier’s fundamental domain $[1988]$.

Finally, we show that, in the case of $k = 1$, $R_{e,\infty}/K_\infty$ corresponds to a face of the Ryshkov polyhedron $R(m) = \{ A \in P_n : m(A) = \min_{0 \neq x \in \mathbb{Z}^n} \langle x, Ax \rangle \geq 1 \}$. For $A \in P_n$, let $S(A)$ denote the set of minimal integral vectors of $A$: 

$$S(A) = \{ x \in \mathbb{Z}^n : m(A) = \langle x, Ax \rangle \}.$$ 

We take $e_1 = t(1, 0, \ldots, 0) \in \mathbb{Z}^n$. It is obvious that the subset $\{ A \in P_n : e_1 \in S(A) \}$ of $P_n$ coincides with $F_{n,1}$. As was shown in [Watanabe 2012, Lemma 1.5], $\mathcal{F}_{\{e_1\}} = F_{n,1} \cap \partial R(m) = \{ A \in F_{n,1} : m(A) = 1 \}$ is a face of $R(m)$. It is easy to see that the map $A \mapsto m(A)^{-1}A$ gives a bijection from $F_{n,1} \cap \text{SL}_n(\mathbb{R})$ onto $\mathcal{F}_{\{e_1\}}$. Therefore, $R_{e,\infty}/K_\infty \cong \pi(R_{e,\infty})$ corresponds to $\mathcal{F}_{\{e_1\}}$.

**Example 4.** Let $k$ be a totally real number field of degree $r$ and $n = 2m$ be an even integer. We consider a symplectic group

$$G(k) = \text{Sp}_n(k) = \left\{ g \in \text{GL}_{2m}(k) : t^g \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \right\}.$$ 

For a fixed $k \in \{1, 2, \ldots, m\}$, let $Q$ denote the maximal parabolic subgroup of $G$ given by

$$Q(k) = U(k)M(k),$$
where
\[
M(k) = \left\{ \delta(a, b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & t_a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} : a \in \text{GL}_k(k), \\
& b = (b_{ij}) \in \text{Sp}_{2(m-k)}(k) \right\}.
\]

\[
U(k) = \left\{ \begin{pmatrix} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{pmatrix} \in G(k) \right\}.
\]

The module of \(k\)-rational characters \(X^*(M)_k\) of \(M\) is a free \(\mathbb{Z}\)-module of rank 1 generated by the character
\[
\hat{\alpha}_Q(\delta(a, b)) = \det a.
\]

The height \(H_Q : G(\mathbb{A}) \rightarrow \mathbb{R}_{>0}\) is given by \(H_Q(g) = |\det a|_\mathbb{A}^{-1}\) if \(g = u\delta(a, b)h\) with \(u \in U(\mathbb{A}), \delta(a, b) \in M(\mathbb{A})\) and \(h \in K\).

We restrict ourselves to the case \(k = m\). An element of \(M(\mathbb{A})\) is denoted by
\[
\delta(a) = \begin{pmatrix} a & 0 \\ 0 & t_a^{-1} \end{pmatrix}, \quad a \in \text{GL}_m(\mathbb{A}).
\]

Let
\[
H_m = \{ Z \in M_m(\mathbb{C}) : ^tZ = Z, \quad \text{Im}Z \in \mathbb{P}_m \}
\]
be the Siegel upper half space and \(H'_m\) the direct product of \(r\) copies of \(H_m\). For \(Z = (Z_\sigma)_{\sigma \in \text{p}_\infty} \in H'_m, \text{Re}Z, \text{Im}Z\) and \(\text{det} Z\) stand for \((\text{Re}Z_\sigma)_{\sigma \in \text{p}_\infty}, (\text{Im}Z_\sigma)_{\sigma \in \text{p}_\infty}\) and \((\text{det} Z_\sigma)_{\sigma \in \text{p}_\infty}\), respectively. The group \(G(k_\infty)\) acts transitively on \(H'_m\) by
\[
gZ = ((a_\sigma Z_\sigma + b_\sigma)(c_\sigma Z_\sigma + d_\sigma)^{-1})_{\sigma \in \text{p}_\infty}
\]
for \(Z = (Z_\sigma) \in H'_m\) and
\[
g = (g_\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}_{\sigma \in \text{p}_\infty} \in G(k_\infty).
\]

The stabilizer \(K_\infty\) of \(Z_0 = (\sqrt{-1}I_m, \ldots, \sqrt{-1}I_m) \in H'_m\) in \(G(k_\infty)\) is a maximal compact subgroup of \(G(k_\infty)\). We choose \(K\) as \(K_\infty \times \prod_{\sigma \in \text{p}_\infty} \text{Sp}_n(\text{o}_\sigma)\). The map \(\pi : g_\infty \mapsto g\{Z_0\}\) from \(G(k_\infty)\) onto \(H'_m\) gives an isomorphism \(G(k_\infty)/K_\infty \cong H'_m\), and hence \(G(k)\backslash G(\mathbb{A})/K \cong G_0\backslash H'_m\). Since \(\text{Im}\{u\delta(a)h\{Z_0\}\} = a^t a\) holds for \(u \in U(k_\infty), a \in \text{GL}_m(k_\infty)\) and \(h \in K_\infty\), we have
\[
H_Q(g_\infty) = \text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}\{g_\infty\{Z_0\}\})^{-1/2} = \left( \prod_{\sigma \in \text{p}_\infty} \det \text{Im}\{g_\sigma\{\sqrt{-1}I_m\}\} \right)^{-1/2}
\]
for any \(g_\infty = (g_\sigma) \in G(k_\infty)\), where \(\text{Nr}_{k_\infty/\mathbb{R}}\) denotes the norm of \(k_\infty\) over \(\mathbb{R}\).
The class number $h_Q$ of $M \cong \text{GL}_m$ defined over $k$ is equal to the class number $h_k$ of $k$. We assume $h_k = 1$ for simplicity. Then $G(k) = Q(k)G_\circ$ and $G(\mathbb{A}) = Q(k)G_{\mathbb{A},\infty}$, and hence

$$m_Q(g_\infty) = \min_{\gamma \in G_\circ} H_Q(\gamma g_\infty).$$

Since

$$\text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}\{\gamma(Z)\}) = \prod_{\sigma \in \mathbb{P}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))|^{-2} \text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}Z)$$

for $Z = (Z_\sigma) \in H^r_m$ and

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_\circ = \text{Sp}_n(o),$$

the condition $m_Q(g_\infty) = H_Q(g_\infty)$ of $g_\infty$ is equivalent with the following condition of $Z = g_\infty(Z_0)$:

$$\prod_{\sigma \in \mathbb{P}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \quad \text{for all} \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_\circ.$$

Therefore, the domain $R_{e,\infty}$ modulo $K_\infty$ is isomorphic to

$$F = \left\{ (Z_\sigma) \in H^r_m : \prod_{\sigma \in \mathbb{P}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \text{ for all } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_\circ \right\}.$$ 

Let $\mathcal{C}$ be an arbitrary fundamental domain of the additive group $M_m(k_\infty)$ with respect to $M_m(o)$, and let $\mathcal{D}$ be an arbitrary fundamental domain of $P^r_m$ with respect to $\text{GL}_m(o)$. It is easy to see that

$$F(\mathcal{C}, \mathcal{D}) = \{ Z \in F : \text{Re}Z \in \mathcal{C}, \text{Im}Z \in \mathcal{D} \}$$

is a fundamental domain of $F$ with respect to $Q_\circ$. By Corollary 22, $F(\mathcal{C}, \mathcal{D})$ is a fundamental domain of $H^r_m$ with respect to $G_\circ$.

If $k = \mathbb{Q}$ and $\mathcal{D}$ is Minkowski’s fundamental domain, then $F(\mathcal{C}, \mathcal{D})$ coincides with Siegel’s fundamental domain [1939].

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